Evaluate an integral

$$I = \int_0^1 \frac{x^{51}}{x^{50} + (1 - x^2)^{25}} dx$$

Let $x = \sin\theta, \quad dx = \cos\theta d\theta$

$$I = \int_0^{\pi/2} \frac{\sin^{51}\theta}{\sin^{50}\theta + (1-\sin^2\theta)^{25}} \cos\theta d\theta = \int_0^{\pi/2} \frac{\sin^{51}\theta\cos\theta}{\sin^{50}\theta + \cos^{50}\theta} d\theta \qquad \text{.... (1)}$$

Let $u = \frac{\pi}{2} - \theta, \quad d\theta = -du$

$$I = \int_{\pi/2}^0 \frac{\cos^{51}u \sin u}{\cos^{50}u + \sin^{50}u} (-du) = \int_0^{\pi/2} \frac{\cos^{51}\theta\sin\theta}{\sin^{50}\theta + \cos^{50}\theta} d\theta \qquad \text{.... (2)}$$

(1) + (2),

$$2I = \int_0^{\pi/2} \frac{\sin^{51}\theta\cos\theta}{\sin^{50}\theta + \cos^{50}\theta} d\theta + \int_0^{\pi/2} \frac{\cos^{51}\theta\sin\theta}{\sin^{50}\theta + \cos^{50}\theta} d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta\cos\theta(\sin^{50}\theta + \cos^{50}\theta)}{\sin^{50}\theta + \cos^{50}\theta} d\theta = \int_0^{\pi/2} \sin\theta\cos\theta \, d\theta = \int_0^{\pi/2} \sin\theta \, d\sin\theta$$

$$= \frac{\sin^2\theta}{2}\Big]_0^{\pi/2} = \frac{1}{2}$$

$$\therefore \ I = \frac{1}{4}$$

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